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The Use of Complex Stiffnesses for Hysteretic Damping

by T. A. Korjack

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Request the following pen-and-ink change be made to Equation 17, page 5, of subject report:

The equation currently reads:

$$F_{r,i} = K_r \frac{R_{r,i} - R_{+1,i}}{\omega_i^2}$$

The equation should read:

$$F_{r,i} = K_r \frac{R_{r,i} - R_{r+1,i}}{\omega_i^2}$$

Abstract

This technical treatise has exemplified a response spectrum method that is particularly suited for spring-mass-damping systems excited by one or more base or ground disturbances characterized by displacements. It offers an alternative to the conventional response spectrum method, which is best suited for a single prescribed ground acceleration. This methodology has also introduced a method of how to incorporate hysteretic damping through complex spring stiffnesses. When the formulation is expressed via base displacements, a simpler application of the response spectrum method to varying spring-mass systems with attendant input excitations can be realized as expressed as the ratio of energy considerations inherent in the natural physics of the phenomena itself.

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1. Introduction

Hysteretic damping is responsible for limiting the oscillations in mechanical systems such as in drive trains of modern tanks. If we look at a linear one-degree-of-freedom spring, mass, and dashpot arrangement, the selection of the proper values of mass, stiffness, and damping constants to produce an overdamped or critically damped system can be effectuated without too much difficulty. If we consider an n-degree-of-freedom spring and mass distributed system model where $X_i(t)$ would be the absolute displacement and $Y(t)$ would be the absolute ground displacement, then it is possible for $Y(t)$ alone to be specified such that $X_i(t)$ may be determined, as in the case of the design of a gas turbine engine so as to resist shock and vibration. The usual classical approach is simply to start out with the laws of motion with respect to the base or foundation framework (Beskos and Boley 1980). The base then becomes fixed where each respective identified mass will be subjected to an inertial force $m_i\ddot{Y}(t)$ and the displacement variables become the relative displacements, viz., $Z_i(t)$.

Hence, we can now proceed with a modal analysis solution using either the acceleration time history, $\ddot{Y}(t)$, or the acceleration response spectrum $A(\omega)$ that is associated with $\ddot{Y}(t)$. Since shock and base line or even ground motion do not have repeatable time histories but do have repeatable shock spectra, then the acceleration response spectrum method will be of greater interest (Biggs 1964). It is also possible to formulate our original speculative problem with a fixed base and with only the mass next to the base subjected to a force equal to $k_nY(t)$ and with the displacement variables being the absolute displacements, $X_i(t)$. What is needed at this point of analysis is to simply cast the equation of motion with respect to an observer at rest and move the term $k_nY(t)$ to the right-hand side of the n^{th} equation.

The purpose of this treatise is to follow through the consequences of this latter formulational ideology such that it can be easily invoked via a suitable computer simulation and then modify the formulation to allow the use of the complex stiffness model of hysteretic damping for oscillations of multi-degree-of-freedom systems, especially inherent in the accessory gear box, reduction gear box, and output shaft of a typical gas turbine engine. This type of analysis can prove to be

invaluable for the diagnostics and prognostics of engines as illustrated by Helfman, Dumer, and Hanratty (1995). This work differs from other investigations by invoking complex stiffnesses for hysteretic damping, whereas in most analyses, the stiffnesses are represented as ordinary spring constants.

2. Concept Formulation

The systems considered herein are those that can be modeled by the matrix differential equation

$$M\ddot{X}(t) + KX(t) = K_n Y(t), \tag{1}$$

where $X(t)$ is an n-dimensional vector of displacements and M and K are nxn symmetric matrices containing the physical parameters of mass and stiffness constants. Let us assume that the eigenvalue problem associated with (1) has a spectral matrix Ω^2 and a modal matrix Φ such that Φ_i is the i[th] eigenvector with elements $\Phi_{r,i}$, where $r = 1, 2, \ldots, n$. Furthermore, also assume that the Φ_i are normalized so that $\underset{r}{MAX}\ \Phi_{r,i} = 1$ for each respective i. In order to have orthogonal conditions, we are required to have

$$\Phi^T M \Phi = M_i \tag{2}$$

and

$$\Phi^T K \Phi = K_i \tag{3}$$

where M_i are the modal masses and K_i are the modal stiffnesses. Furthermore, it can be easily shown that $\Omega_i^2 = Ki/M_i$. If we look at the transformation of coordinates (Brent 1973), viz.,

$$X(t) = \Phi q(t), \tag{4}$$

then (1) becomes,

$$M\Phi\ddot{q}+K\Phi q=K_nY. \tag{5}$$

Then, if we premultiply by Φ^T and use (2) and (3), we then have,

$$M_i\ddot{q}+K_iq=\Phi^TK_nY. \tag{6}$$

Also, it can be shown that

$$\Phi^T = \Phi_{n,i}, \tag{7}$$

i.e., a vector composed of the n^{th} element of each Φ_i. Then, (5) becomes

$$M_i\ddot{q}+K_iq_i = \Phi_{n,i}K_nY. \tag{8}$$

The response spectrum is the maximum response of an oscillator subjected to a forcing function drawn from the process of interest as a function of the frequency of the oscillator. If the response of interest is the absolute acceleration, then the acceleration response spectrum is denoted by A(ω), where ω is the angular frequency of the oscillator. If

$$A(\omega_i) = \underset{t}{MAX}\ \ddot{Q}_i(t), \tag{9}$$

where $Q_i(t)$ is the solution to

$$M_i\ddot{Q} + K_iQ_i = K_iY, \tag{10}$$

then, linearity requires that

$$\underset{t}{MAX}\ \ddot{q}_i(t) = \Phi_{n,i}A(\omega_i)\frac{K_n}{K_i}\ , \tag{11}$$

where $q_i(t)$ is the solution to (8). Furthermore, (4) can be recast as (Byrne and Hall 1973)

$$\ddot{X}_r = \sum_{i=1}^{n} \Phi_{r,i} \ddot{q}_i . \tag{12}$$

The acceleration of the rth degree-of-freedom in the ith mode can therefore be expressed as

$$\ddot{X}_{r,i} = \Phi_{r,i} \ddot{q}_i \ , \tag{13}$$

and using (11), we then have

$$\underset{t}{MAX}\ \ddot{X}_{r,i} = \Phi_{r,i} \Phi_{n,i} A(\omega_i) \frac{K_n}{K_i} . \tag{14}$$

Also, if we let (Inman and Andry 1980)

$$\underset{t}{MAX}\ \ddot{X}_{r,i} = R_{r,i} \ , \tag{15}$$

then, in going from (13) to (14), the information about the temporal relationships among the terms on the left-hand side of (15) will be lost. This loss of information requires that the exact modal recombination procedure, viz. (4), be suitably replaced by an approximate modal recombination rule. If we use the square root of the sum of the squares, i.e.,

$$A_r = \sqrt{\sum_{i=1}^{n} R_{r,i}^2} \ , \tag{16}$$

then A_r is expected to be an approximate measure of the maximum absolute acceleration of mass point *Mr* when all modes are excited by input *Y(t)*. The spring forces in mode *i* can be obtained from

$$F_{r,i} = K_r \frac{R_{r,i} - R_{+1,i}}{\omega_i^2} , \tag{17}$$

and, if required, they can also be combined by the square root of the sum of the squares rule. Equations (14) and (17) can also be the basis for a computer solution of (1) using the acceleration response spectrum associated *Y(t)*. If $A(\omega)$ is simply not specified, a separate program will be necessary to compute $A(\omega)$ from *Y(t)* in accordance with (10). In the case of multiple, distinct inputs, as is the case in the structure of the gas turbine engine, it is better to generalize (1), as done similarly by Biggs (1964) and base the solution on the prescribed displacements.

3. Effect of Damping

In lumped spring-mass models subjected to transcendental forcing such as sinusoidal, which occurs within many electrical components as in a gas turbine engine, hysteretic damping can be included by considering the spring stiffnesses to be of the complex mode (Inman and Andry 1980), i.e., $K^* = K_R + iK_I$, where the superscript * represents a complex number, the subscript *r* is the real part of the complex number, and the subscript *I* is the imaginary part. This representation is a mathematical convenience rather than a physical depiction as are complex numbers themselves. In fact, hysteretic damping is energy dissipation, which is independent of frequency and is proportional to the square of the displacement. Hence, hysteretic damping is a much better predictor of the ability of metals and polymers to dissipate energy than is linear damping. Shock response spectra are usually specified with damping as a parameter. If we are to use (14), we should select the value of $A(\omega_i)$ for the appropriate damping value. If damping is measured by the loss factor given as η, then (14) can be expressed as

$$R^*_{r,i} = \Phi_{r,i}\Phi_{n,i}A(\omega_i,\eta_i)\frac{K^*_n}{K_i} , \tag{18}$$

where K_i can be given as,

$$K_i = \Phi_i^T K_R \Phi_i. \tag{19}$$

We are now left with the question of how to find η_i for each respective mode of the vibrational spectrum. In order to adequately address this question, we should first consider the simple case of a single-degree-of-freedom system subjected to sinusoidal forcing. We are then led to the well-known equation of motion as,

$$m\ddot{x}^* + k^* x^* = f^*, \tag{20}$$

where

$$f(t) = Re\left[f^* e^{iwt}\right], \tag{21}$$

$$x(t) = Re\left[x^* e^{iwt}\right]. \tag{22}$$

If we let the loss factor to be defined as

$$\eta = \frac{K_I}{K_R}, \tag{23}$$

then (20) simply becomes,

$$M\ddot{X}^* + K_R(1 + i\eta)X^* = f^*. \tag{24}$$

Hence, we can now proceed to an n-degree-of-freedom system such that

$$M\ddot{X}^* + iK_i X^* + K_R X^* = F^*, \tag{25}$$

such that

$$F^* = K_n^* Y, \tag{26}$$

and

$$Y(t) = Re\left[Y^{*e^{i\omega t}}\right]. \tag{27}$$

If we desire a generalization of (4) such that (Inman and Andry 1980)

$$X^*(t) = \Phi q^*(t), \tag{28}$$

where Φ is the real modal matrix derived from the undamped eigenvalue problem, then we are led to the following i[th] degree-of-freedom relationship of (25) by using (2) and (3), i.e.,

$$M_i \ddot{q}_i^* + K_i q^* + i\Phi_i^T K_I \Phi_i q_i^* = \Phi_i^T F_i^*. \tag{29}$$

Hence, if we compare (29) and (24), a modal loss factor can be suitably distinguished as

$$\eta_i = \frac{\Phi_i^T K_I \Phi_i}{\Phi_i^T K_R \Phi_i}, \tag{30}$$

such that we are assuming that the damping is small so that no significant errors will arise from using the undamped mode shapes. We can write (30) in another form, such as

$$\eta_i = \frac{\sum_{j=1}^{n} (K_j)_I (\Phi_{i,j} - \Phi_{i,j+1})^2}{\sum_{j=1}^{n} (K_j)_R (\Phi_{i,j} - \Phi_{i,j+1})^2}, \tag{31}$$

where we have the imaginary and complex part of the complex stiffness. If we further let

$$\left(K_j\right)_I = \eta_j \left(K_j\right)_R , \tag{32}$$

in such a way that (31) is transformed into

$$\eta_i = \frac{\sum_{j=1}^{n} \eta_j \left(K_j\right)_R \left(\Phi_{i,j} - \Phi_{i,j+1}\right)^2}{\sum \left(K_j\right)_R \left(\Phi_{i,j} - \Phi_{i,j+1}\right)^2}, \tag{33}$$

then it becomes clear that (33) is the ratio of potential energies, i.e., the numerator is the sum of the energies dissipated in each spring in mode i, and the denominator is the sum of the energies stored in each spring in mode i. Hence, this equation should be able to be used to estimate modal loss factors, since it can be justified merely on the basis of natural intuitive estimation. If we look at the relationship between (33) and (30), we are formally led to the conclusion that (33) is essentially the modal superposition using complex stiffnesses.

4. Conclusion

This technical treatise has exemplified a response spectrum method that is particularly suited for spring-mass-damping systems excited by one or more base or ground disturbances characterized by displacements. It offers an alternative to the conventional response spectrum method, which is best suited for a single prescribed ground acceleration. This methodology has also introduced a method to incorporate hysteretic damping through complex spring stiffnesses. The response spectrum method was formulated in terms of base displacements rather than simple base acceleration, which is characteristic of base shear calculations for building and structural vibration analysis under seismic and ground disturbances or artillery base loads. When the formulation is expressed via base displacements, a simpler application of the response spectrum method to varying spring-mass systems with attendant input excitations can be realized, as expressed as the ratio of energy considerations inherent in the natural physics of the phenomena itself.

Hence, the energy ratio equation should be able to be used to estimate modal loss factors, since it can be justified merely on the basis of natural intuitive estimation. Furthermore, the use of this derived relation might be able to be applied to handle single frequency excitations that occur in a typical gas turbine engine for diagnostics involving component failures or even routine maintenance. This treatise has identified the original, practical use of simulating complex stiffnesses for hysteretic damping, which is quite different from traditional representations of the usual spring-mass-damping phenomenology.

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